Hadron structure from nonperturbative QCD: the quark propagator DSE

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Abstract. The quark propagator (S) is a useful quantity in hadron physics as it gives us, for example, the momentum dependence of the quark mass function. S is a function of p, and for low momenta a nonperturbative treatment is required to obtain it, *e.g.*, solving the quark's Dyson-Schwinger equation (as done in this project). It is at this low-momentum / strong interaction regime that the quark gains a significant mass, due to the gluonic interactions. Evaluating $S(p^2)$ in the complex plane leads to singularities, which are useful in further hadron physics calculations and can be interpreted as signatures of confinement.

KEYWORDS: Quark Propagator, DSE, $M(p^2)$, Renormalization-Group Invariance, Chiral Limit, Quark Condensate, $p^2 \in \mathbb{C}$, Poles, Schlessinger Method

1 Introduction

A hadron is a composite particle made of two or more quarks and/or antiquarks bonded by the strong force. QCD (Quantum Chromodynamics) is the theory that describes the strong interaction. While we can treat QCD perturbatively when the momenta is large, using just a reduced number of terms in a perturbative series; in hadrons, the quark momenta are small, because they are interacting a lot, and, because the coupling becomes strong at low momenta, we cannot do the perturbative approximation. Hence, *Hadron Structure from Nonperturbative QCD*.

1.1 Quark Propagator - Components

The quark propagator $S(p)$ is a fundamental quantity in QCD which is also used in hadron physics calculations. We can define $S(p)$ in terms of its dressing functions, that is, we can define its components. Not surprisingly, there are infinitely many different possible sets of two dressing functions / components. For example, Eqs. (3.1) or (A.1) in [1]:

$$S(p^2) = -i\not{p}\sigma_v(p^2)+\sigma_s(p^2) = \frac{Z_f(p^2)}{p^2 + [M(p^2)]^2}(-i\not{p}+M(p^2)).$$
(1)

Above, the first set of two dressing functions (σ_v and σ_s) is the simplest; but the second one (Z_f and M) gives us the quark mass function $M(p^2)$ which encodes the quark's mass. Different dressing functions can have different properties and meanings, as we will see later. It is important to notice that S is a function of p (the four-vector p^μ), and so are its dressing functions (*e.g.*, $M(p^2)$, with p^2 being a Lorentz invariant one can do with p^μ). Thus, we have a mass that is a function of the quark's momentum. The quark mass gets larger for low momenta as a result from the strong interaction. The results in Fig.1 show us this dependency explicitly.
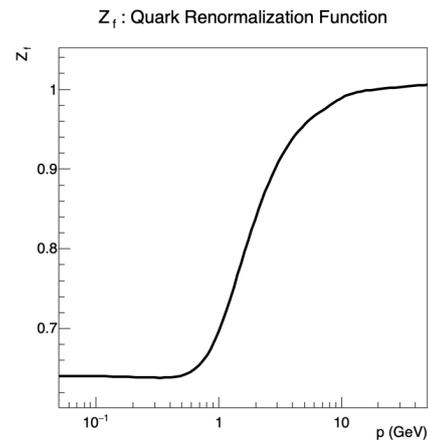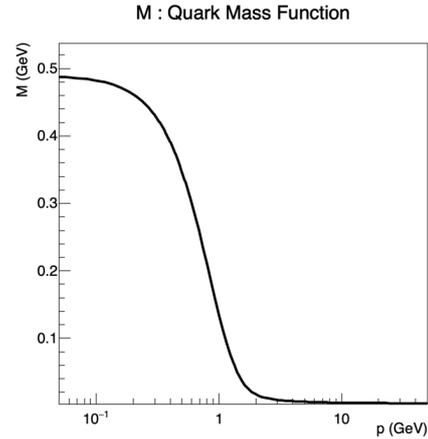
Figure 1: Quark propagator dressing functions *vs.* momentum (for a light quark - input point: $\mu = 19$ GeV, $M(\mu) = 4$ MeV)

These results were part of this project's goals. We can see how things change a lot for the low-momentum / nonperturbative regime. In perturbation theory, we would

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only capture the large-momentum tails of these quantities, hence the importance of a nonperturbative treatment.

In a way, we can make an analogy between the quark's mass *vs.* its momentum and a ferromagnet's magnetization *vs.* its temperature. Below a critical temperature it gains permanent magnetization, as a quark gains mass below a critical momentum. To make the analogy better, we see that above that turning point in momentum, we still have a mass (the current quark mass) - which in this analogy would be a background magnetic field. (Nevertheless, this is no more than an analogy.)
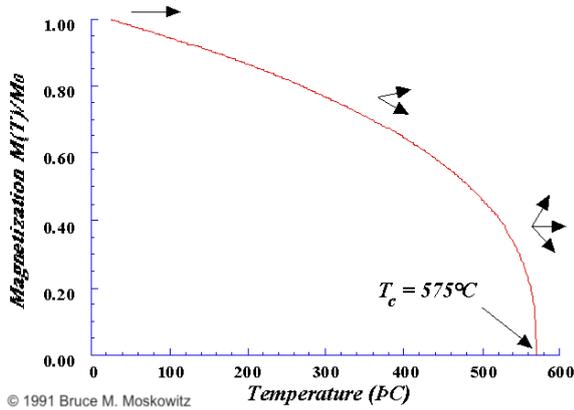
Figure 2: Analogy - ferromagnet's magnetization *vs.* temperature (Ref.[2])

1.2 Quark Propagator - DSE

We already know the decomposition of S in terms of its components, which contain the information it has. Now we need a way to determine them through a dynamical equation.

We will use the quark's Dyson-Schwinger equation (DSE), (A.6) in [1]:

$$S^{-1} = Z_2(i p\!\!\!/ + M_\Lambda) + \Sigma. \qquad (2)$$

In a pictorial way, where the l.h.s represents $S(p)^{-1}$, it reads:
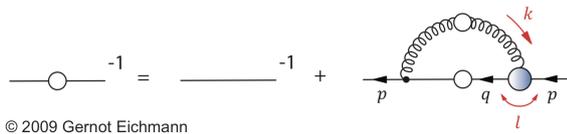
Figure 3: Dyson-Schwinger equation (Ref.[1])

It is worth mentioning that the quantity we are looking for, the quark propagator, reappears on the r.h.s. This suggests to solve the equation iteratively.

The new symbols in Eq. (2) are given by:

$$Z_2 = Z_f^{-1}|_{p^2=\Lambda^2}, \qquad M_\Lambda = M|_{p^2=\Lambda^2}, \qquad (3)$$

where Λ is the ultra-violet regularization scale. Basically they are just the values of the dressing functions at a certain momentum.

Σ is the quark self-energy. In its pictorial representation (last term), one can see the quark propagator on the bottom (the circle with a straight line) and a new propagator on top (the circle with a spring-like line). This is the gluon propagator. There is also a gluon DSE, but in this project we used a model for the gluon instead of solving a coupled equation. Finally, the blue circle is the quark-gluon vertex, which is also modeled.

In a nutshell, the DSE describes all the ways how a quark can emit and absorb gluons.

2 Quark DSE on the Real Axis

In this project, we used Fortran90 for computation and ROOT for graphics. The computation was constructed around an iterative method until convergence, where in each iteration there were 2 steps:

1. Calculate the integrals for Σ
   $\rightarrow$ Update the M and Z_f functions.

2. Affirm the input point ($p = \mu, M|_\mu = m, Z_f|_\mu = 1$)
   $\rightarrow$ Update the Z_2 and M_Λ numbers.

This results in Fig.1 and similar ones.

For more information on the DSE and how it was solved, see appendix A.

2.1 Renormalization-Group Invariance

One of the things one can test right away at this point is the following:

1. Give a certain input point ($p = \mu, M|_\mu = m, Z_f|_\mu = 1$) and let the routines calculate the dressing functions.

2. From the obtained mass function get the value of M at a μ', calling it m'.

3. Using this new input point ($p = \mu', M|_\mu = m', Z_f|_\mu = 1$), run the routines again and compare the obtained dressing functions with the original ones.
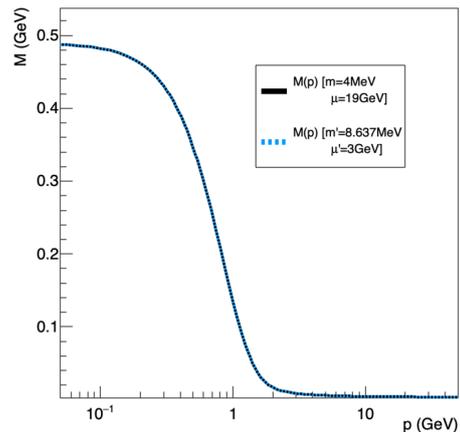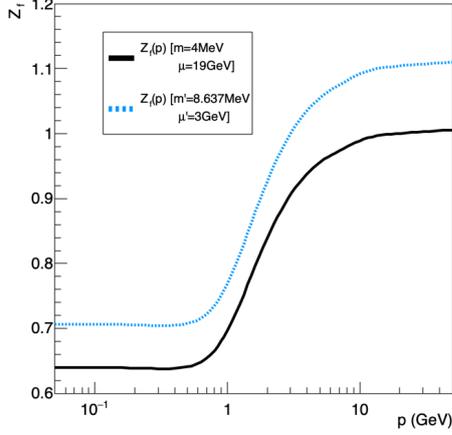
Figure 4: Renormalization-group invariance for $M(p^2)$, but not for $Z_f(p^2, \mu^2)$

What we see in the results above is that for the mass dressing function the lines are coincident, but for the Z_f dressing function they are not. So one can say that Z_f is a function also of μ^2, not only of p^2. This is one of the properties we mentioned that well chosen dressing functions can have. For example, for the pair σ_v and σ_s, neither of those have this kind of invariance since both depend on μ^2.

2.2 Chiral Limit - Quark Condensate

Working in the *chiral limit* can be implemented by setting our renormalization-point dependent mass (as long as μ is large) to 0. This allows for the calculation of the quark condensate, the expectation of the vacuum state $< 0|\bar{\Psi}\Psi|0 >$, which breaks chiral symmetry. (For details, see appendix B.)

Table 1: Quark condensate values for different values of η, a parameter used in the gluon model

η	$-<\bar{q}q>$
1.60	$(219.996 \text{ MeV})^3$
1.80	$(220.078 \text{ MeV})^3$
2.00	$(219.406 \text{ MeV})^3$

3 Quark DSE in the Complex Plane

Until now, we have been calculating the quark propagator dressing functions for a real momentum, more precisely, for a real momentum squared. But what happens if we calculate those quantities for a complex momentum squared?

We obtain this in two steps:

1. Let the DSE converge for $p^2 \in \mathbb{R}$, just like we have done before.

2. Calculate M and Z_f through the DSE, now with $p^2 \in \mathbb{C}$. (Without iterating, the system has already converged; and only using (5), not (6).)
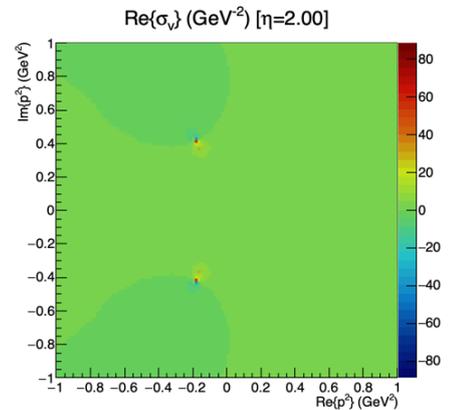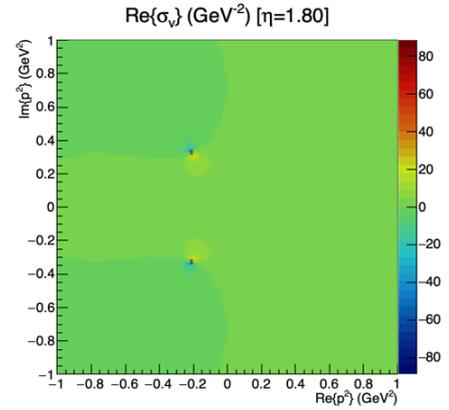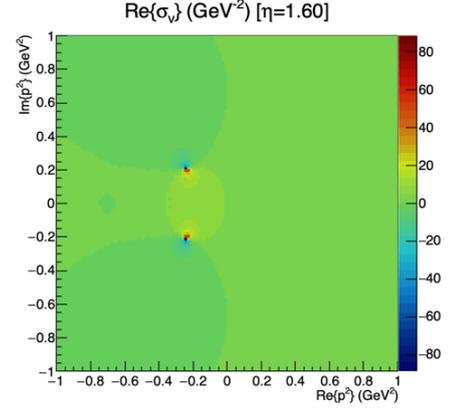
Figure 5: Quark propagator for $p^2 \in \mathbb{C}$ - direct calculation through the DSE

Fig.5 shows the real part of the σ_v dressing function for different values of η (a parameter used in our quark self-energy model). One can see that there are poles, and that those poles move around for different values of η. In this way, one can study the properties of the model inputs through this method of seeing the quark propagator in the complex momentum plane. Another thing one can do is get the precise position of the poles and their residues[1];

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[1]In this project, we have done so successfully and we are planing to do it with other methods to find out the best one, but we do not include the results in this report.

both quantities are useful in further hadron physics calculations.

One really interesting subject is that of confinement. A free particle implies a pole for $\mathbb{R}^-$ (euclidean momenta $p^2 < 0$). Here we do not find this, so we do not have a free particle. The definition of confinement and its study through the analytic structure of a particle's propagator is an open problem.

4 Get f($p^2 \in \mathbb{C}$) from f($p^2 \in \mathbb{R}$)

In the last section we calculated M and Z_f through the DSE for $p^2 \in \mathbb{C}$, assuming this can be done in a straight forward way. Our DSE used relatively simple model inputs for which this is indeed possible. But for other models this is not the case, since we cannot do a direct substitution with a complex momentum.[2] However, we can always solve the DSE for $p^2 \in \mathbb{R}$. Can we then extrapolate a function with known values on the positive real axis to the complex plane?

Of course, there are mathematical techniques to do so. For example, the Schlessinger Point Method [3] uses n input points (z_i, F_i), which can all be in $\mathbb{R}^+$, Eq. (30) in [4]:

$$f(z) = \cfrac{c_1}{1 + \cfrac{c_2(z-z_1)}{1 + \cfrac{c_3(z-z_2)}{\dots}}}. \qquad (4)$$

We just have to determine n parameters c_i and then we can reconstruct an analytic continuation f of the original function. Both can be done with iterative methods (see appendix C).

Figure 6: Quark propagator on $p^2 \in \mathbb{C}$ - Schlessinger Point Method

Again, one can get the position and residue of each pole and compare with those of the direct calculation[3] (for models where such a calculation is possible) to check the reliability of this extrapolation method.

5 Conclusions and Future Work

Using analytic continuation methods, we have achieved a practical way of solving the quark DSE for any gluonic input. Having the quark propagator, in principle we can use the dressing functions of interest in hadron physics calculations for experimentally testable observables, test mathematical properties, obtain other quantities of interest, *etc*. Also, we established both a method to directly obtain S in the complex plane and a purely extrapolative one. With this we obtained the pole positions and residues. These poles move for different gluon input models; analysing those movements is a way of testing the models.

There is a list of things we plan to do next:

- Schlessinger method: test the number and distribution of input points and their implications for the method's reliability (using the direct method as a comparison).

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[2]For other gluon models, the expression for $\alpha(k^2)$ (for our model we used Eq. (8)) the direct complex p^2 substitution is impossible, because there appears a pole in the integrand of (7). Since it is a double integration, after the first integration the pole will produce to a branch cut, which complicates the second integration. This issue is related with the dependency of $\alpha(k^2)$ on certain powers of k.

[3]We did this and found that the results from the different methods matched within the error margin. The values are not shown, since they are of limited relevance.

- Use other models for the gluonic ingredients.

- Explore machine learning as an alternative to the Sch-lessinger method, using the results from the direct method to learn how to extrapolate.

Acknowledgements

A Solving the Quark DSE

Here we explain in more depth the DSE, the employed models and the computation. Throughout this appendix, we refer to the equivalent equation numbers in Ref.[1].

The quark DSE was presented in (2). It can be rewritten in terms of two coupled integral equations for two dressing functions. We did not use either of the sets presented in (1) for actual computations; instead we used M and A (A is simply $1/Z_f$). Eqs. (A.7) in [1] read:

$$A(p^2) = Z_2 + \Sigma_A(p^2),$$
$$M(p^2)A(p^2) = M_\Lambda Z_2 + \Sigma_M(p^2), \quad (5)$$

where $\Sigma = i\slashed{p}\Sigma_A + \Sigma_M$.

As mentioned in section 2, in each iteration we first calculate the integrals for Σ, update the dressing functions, and then affirm our input point. So we calculate those integrals (Eqs. (7)) and then put them in the equations above to update the dressing functions. To affirm our input point ($p^2 = \mu^2, M|_\mu = m, Z_f|_\mu = 1$), that is, our renormalization condition with m as an input parameter, we write:

$$A(\mu^2) = 1, \qquad M(\mu^2) = m.$$

We use (A.9) in [1], which follows straight out of Eqs. (5):

$$Z_2 = 1 - \Sigma_A(\mu^2),$$
$$M_\Lambda = \frac{M(\mu^2) - \Sigma_M(\mu^2)}{Z_2} = \frac{m - \Sigma_M(\mu^2)}{1 - \Sigma_A(\mu^2)}. \quad (6)$$

* * *

Finally, we collect the integrals associated with the quark self-energy. With Λ being the cutoff in the system ($p_{max}^2 = \Lambda^2$), we have (A.11) in [1]:

$$\Sigma_A(p^2) = \int_q^\Lambda \sigma_v(q^2)g(k^2)F(p^2, q^2, z),$$
$$\Sigma_M(p^2) = 3\int_q^\Lambda \sigma_s(q^2)g(k^2). \quad (7)$$

We use the set of dressing functions $\{\sigma_v, \sigma_s\}$ in these integrals. To write them in terms of M and A, we have $Z_f(q^2) = 1/A(q^2)$ and from (1) it follows that:

$$\sigma_v(q^2) = \frac{Z_f(q^2)}{q^2 + [M(q^2)]^2}, \quad \sigma_s(q^2) = \frac{Z_f(q^2)M(q^2)}{q^2 + [M(q^2)]^2}.$$

The dimensionless quantity F is given by (A.12) in [1]:

$$F(p^2, q^2, z) = \frac{-k^2 + \frac{p^2+q^2}{2} + \frac{(p^2-q^2)^2}{2k^2}}{p^2},$$

where k is the gluon momentum with

$$k^2 = p^2 + q^2 - 2\sqrt{p^2}\sqrt{q^2}z.$$

The quantity $g(k^2)$ stands for the effective coupling, (A.10) in [1]:

$$g(k^2) = Z_2^2 \frac{16\pi}{3}\frac{\alpha(k^2)}{k^2}.$$

* * *

Because we are not solving the DSEs for the gluon propagator and quark-gluon vertex, we employ a model for these quantities, that is, a model parametrization for $\alpha(k^2)$. We use the Maris-Tandy interaction [5], as in Eq. (3.96) of Ref.[6]:

$$\alpha(k^2) = \pi\eta^7 x^2 e^{-\eta^2 x} + \frac{2\pi\gamma_m(1 - e^{-k^2/\Lambda_t^2})}{ln[e^2 - 1 + (1 + k^2/\Lambda_{QCD}^2)^2]}, \quad (8)$$

with:

- $\Lambda_t = 1$ GeV,
- $\Lambda_{QCD} = 0.234$ GeV,
- $\gamma_m = 12/25$,
- $x = k^2/\Lambda^2$,
- $\Lambda = 0.72$ GeV is an infrared scale (*i.e.*, not the cutoff in the system),
- $\eta = 1.6...1.8...2.0$ is a dimensionless parameter to which many observables are insensitive in the range presented.

The first term in Eq. (8) is α_{IR} and the second α_{UV}. For the momentum regime we are interested (low-momentum): $\alpha_{IR} >> \alpha_{UV}$; but we keep both terms in the calculations anyway.

* * *

To perform the integrals in Eqs. (7), in Ref.[1], see Eq. below (A.12) and Eq. (B.6):

$$\int_q^\Lambda \sigma_v gF = \frac{2\pi}{(2\pi)^4}\int_0^{\Lambda^2} dq^2 q^2 \sigma_v(q^2)$$
$$\int_{-1}^1 dz\sqrt{1 - z^2}g(k^2)F(p^2, q^2, z).$$

Making the variable change $\frac{d(q^2)}{dq} = 2q$, we obtain the final expression that we used in the code:

$$\int_q^\Lambda \sigma_v gF = \frac{4\pi}{(2\pi)^4} \int_0^\Lambda dq q^3 \sigma_v(q^2)$$
$$\int_{-1}^1 dz \sqrt{1-z^2} g(k^2) F(p^2, q^2, z).$$

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In each iteration, we first updated our dressing functions and then updated the numbers Z_2 and M_Λ. Using the start guess for those numbers:

$$Z_2 = 1, \qquad M_\Lambda = 1, \qquad (9)$$

and for the dressing functions:

$$A(p^2) = Z_2, \qquad M(p^2) = M_\Lambda Z_2 / A(p^2). \qquad (10)$$

With this, we have everything needed to iterate and solve the quark DSE.

B Chiral Limit - Quark Condensate: Details

The chiral limit is defined by setting the renormalization-point independent current mass to zero ($\hat{m} = 0$). Since we use a cutoff Λ^2 for p^2, we can implement this by $m_0(\Lambda^2) = 0$ (m_0: cutoff-dependent bare current-quark mass).

With Z_m the mass renormalization constant and m_μ the renormalized mass:

$$m_0(\Lambda^2) = Z_m(\mu^2, \Lambda^2) m_\mu = \frac{M(\Lambda^2)}{M(\mu^2)} m_\mu.$$

For a large renormalization point, $\mu \gg \Lambda_{QCD}$ (e.g., $\mu = 19$ GeV), this implies $M(\mu^2) = m_\mu$. Using this in the equation above entails $m_0(\Lambda^2) = M(\Lambda^2)$.

This means that, for $\mu \gg \Lambda_{QCD}$, the *chiral limit* corresponds to $M(\Lambda^2) = 0$, or, using the notation in Eq. (3), $M_\Lambda = 0$.

* * *

The quark mass function M obtained with the DSE asymptotically behaves as predicted in perturbation theory (valid for large momenta), (A.3) in [1]:

$$M(p^2) \xrightarrow{p^2 \to \infty} \frac{\hat{m}}{\mathcal{F}(p^2)^{\gamma_m}} + \frac{2\pi^2 \gamma_m}{N_C} \frac{-<\bar{q}q>}{\mathcal{F}(p^2)^{1-\gamma_m} p^2},$$

where:

- $\mathcal{F}(p^2) = \frac{1}{2} ln(p^2/\Lambda_{QCD}^2)$,
- $N_C = 3$ is the color trace,
- $-<\bar{q}q>$ is the quark condensate.

For a non-zero $\hat{m}$, the second term is suppressed by $1/p^2$. For $\hat{m} = 0$, only the second term remains; so the quark condensate can be obtained, using a large value for p^2 (e.g., $p^2 = (10^3$ GeV$)^2 = \Lambda^2$), from the following equation:

$$-<\bar{q}q> \approx \frac{N_C}{2\pi^2 \gamma_m} \mathcal{F}(p^2)^{1-\gamma_m} p^2 M(p^2). \qquad (11)$$

* * *

Summing up, to obtain the quark condensate through the chiral limit, we just have to set $M_\Lambda = 0$ in (6) when solving the DSE; and in the end calculate (11).

Notice that if one sets $M_\Lambda = 0$ in the start guess (Eqs. (9)), the start guess for M will be zero (Eqs. (10)), making it stuck there. So our start guess for M_Λ is still 1, but then we affirm $M_\Lambda = 0$ in every iteration.

Note: The chiral limit can also be used as an approximation for light quarks, which have a small current-quark mass (e.g., $M(\mu = 19$ GeV$) = 4$ MeV).

C Schlessinger Method - Iterative Algorithms

The parameters c_i in the Schlessinger Point Method, Eq. (4), are determined by:

$$c_1 = F_1, \qquad c_2 = \frac{(c_1/F_2) - 1}{z_2 - z_1},$$

$$c_i = \frac{\left[\frac{c_j(z_i - z_{j-1})}{\frac{c_1}{F_i} - 1} - 1 \right]}{z_i - z_{i-1}}, i > 2,$$

with the blue "operator" repeated $i - 2$ times over itself ($j = 2...i - 1$).

* * *

For the f reconstruction at $z \in \mathbb{C}$, we write

$$f(z) = \frac{c_1}{\left[1 + \frac{c_j(z - z_{j-1})}{1 + c_n(z - z_{n-1})} \right]},$$

with the blue "operator" repeated $n - 2$ times over itself ($j = n - 1...2$).

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